

# SHARPNESS OF UNIFORM CONTINUITY OF QUASICONFORMAL MAPPINGS ONTO $s$ -JOHN DOMAINS

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ABSTRACT. We construct examples to show the sharpness of uniform continuity of quasiconformal mappings onto  $s$ -John domains. Our examples also give a negative answer to a prediction in [7].

## 1. INTRODUCTION

Recall that a bounded domain  $\Omega \subset \mathbb{R}^n$  is a John domain if there is a constant  $C$  and a point  $x_0 \in \Omega$  so that, for each  $x \in \Omega$ , one can find a rectifiable curve  $\gamma : [0, 1] \rightarrow \Omega$  with  $\gamma(0) = x$ ,  $\gamma(1) = x_0$  and with

$$(1.1) \quad Cd(\gamma(t), \partial\Omega) \geq l(\gamma([0, t]))$$

for each  $0 < t \leq 1$ . F. John used this condition in his work on elasticity [8] and the term was coined by Martio and Sarvas [10]. Smith and Stegenga [12] introduced the more general concept of  $s$ -John domains,  $s \geq 1$ , by replacing (1.1) with

$$(1.2) \quad Cd(\gamma(t), \partial\Omega) \geq l(\gamma([0, t]))^s.$$

The recent studies [1, 5, 6] on mappings of finite distortion have generated new interest in the class of  $s$ -John domains.

In this paper, we are interested in uniform continuity of those quasiconformal mappings whose target domain is  $s$ -John. For the  $s = 1$  case, one always has uniform Hölder continuity:

$$(1.3) \quad |f(x) - f(y)| \leq Cd_I(x, y)^\alpha,$$

provided  $f : \Omega' \rightarrow \Omega$  is a quasiconformal mapping, see [9]. Here  $\alpha$  depends on the constant in the 1-John condition for  $\Omega$ , on the quasiconformality constant of  $f$ , and on the underlying dimension. The internal distance  $d_I(z, w)$  for a pair of points in a domain  $G$  is the infimum of the lengths of all paths that join  $z$  to  $w$  in  $G$ .

In [4], the following uniform continuity result for general  $s$ -John domains was established.

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**Theorem 1.1.** *Let  $\Omega' \subset \mathbb{R}^n$  be a domain and  $\Omega \subset \mathbb{R}^n$  be an  $s$ -John domain with  $s \in (1, 1 + \frac{1}{n-1})$ . Then each quasiconformal mapping  $f : \Omega' \rightarrow \Omega$  satisfies*

$$(1.4) \quad D_I(f(x'), f(y')) \leq C \left( \log \frac{1}{C d_I(x', y')} \right)^{-\frac{1}{s-1}}$$

for every pair  $x', y'$  of distinct points in  $\Omega'$ , where  $D_I$  is defined by taking the infimum of the diameters over all rectifiable curves in  $\Omega$  joining the desired pair of points.

Notice that for all  $z, w \in \Omega$ ,

$$|z - w| \leq D_I(z, w);$$

hence the left-hand side of (1.4) can be replaced with  $|f(x') - f(y')|$ . It was shown in [4] that the restriction  $s < 1 + \frac{1}{n-1}$  can be disposed of if  $\Omega$  equipped with the quasihyperbolic metric is Gromov hyperbolic.

Our first result of this paper shows that the requirement that  $s < 1 + \frac{1}{n-1}$  in Theorem 1.1 is sharp when  $n = 2$ . This is somewhat surprising since (1.4) does not degenerate when  $s = 1 + \frac{1}{n-1}$ .

**Theorem 1.2.** *There exist a bounded domain  $\Omega' \subset \mathbb{R}^2$ , a 2-John domain  $\Omega \subset \mathbb{R}^2$ , and a quasiconformal mapping  $f : \Omega' \rightarrow \Omega$  such that  $f$  is not uniformly continuous with respect to the metrics  $d(x, y) = |x - y|$  in  $\Omega$  and  $d_I$  in  $\Omega'$ .*

Since each simply (or finitely) connected planar domain is Gromov hyperbolic when equipped with the quasihyperbolic metric, the 2-John domain above is necessarily infinitely connected.

Actually, the results in [9] establish (1.3) under more general setting than the case of 1-John domains. Indeed, it was proven there that it suffices to assume that

$$(1.5) \quad k_\Omega(x, x_0) \leq C_1 \log \frac{1}{d(x, \partial\Omega)} + C_2$$

for some constants  $C_1, C_2$  and a fixed point  $x_0 \in \Omega$ , where

$$k_\Omega(x, x_0) = \inf_{\gamma_x} \int_{\gamma_x} \frac{ds}{d(z, \partial\Omega)}$$

is the quasihyperbolic distance between  $x$  and  $x_0$ ; the infimum is taken over all rectifiable curves  $\gamma_x$  in  $\Omega$  which join  $x$  to  $x_0$ . For  $x, y \in \Omega$ , there is a (quasihyperbolic) geodesic  $[x, y]$  in  $\Omega$  with

$$k_\Omega(x, y) = \int_{[x, y]} \frac{ds}{d(z, \partial\Omega)},$$

see [3]. In [7], (1.5) was further replaced with

$$k_\Omega(x, x_0) \leq \phi \left( \frac{1}{d(x, \partial\Omega)} \right),$$

under the assumption that

$$\int_1^\infty \frac{dt}{\phi^{-1}(t)} < \infty.$$

A uniform continuity estimate of the type (1.3) was established under the additional assumption that  $t \mapsto \Phi(t)^{-a}$  is concave for some  $a > n - 1$ , where

$$\Phi(t) = \psi^{-1}(t) \quad \text{and} \quad \psi(t) = \int_t^\infty \frac{ds}{\phi^{-1}(s)}.$$

This concavity assumption was speculated in [7] to be superfluous. Our construction refutes this speculation.

**Corollary 1.3.** *There exist a bounded domain  $\Omega \subset \mathbb{R}^2$  and a point  $x_0 \in \Omega$  such that*

$$(1.6) \quad k_\Omega(x, x_0) \leq C d(x, \partial\Omega)^{-\frac{1}{2}}$$

*for all  $x \in \Omega$  and a quasiconformal mapping  $f : \Omega' \rightarrow \Omega$ , where  $\Omega' \subset \mathbb{R}^2$ , such that  $f$  is not uniformly continuous with respect to the metrics  $d(x, y) = |x - y|$  in  $\Omega$  and  $d_I$  in  $\Omega'$ .*

Our next result shows that Theorem 1.1 is essentially sharp for all dimensions.

**Theorem 1.4.** *Let  $n \geq 3$ . There exist a bounded domain  $\Omega' \subset \mathbb{R}^n$ , a domain  $\Omega \subset \mathbb{R}^n$  that is  $s$ -John for any  $s \in (1 + \frac{1}{n-1}, \infty)$ , and a quasiconformal mapping  $f : \Omega' \rightarrow \Omega$  such that  $f$  is not uniformly continuous with respect to the metrics  $d(x, y) = |x - y|$  in  $\Omega$  and  $d_I$  in  $\Omega'$ .*

It would be interesting to know whether one can allow for  $s = 1 + \frac{1}{n-1}$  in Theorem 1.4.

As a by-product of our construction, we obtain the following corollary, which implies that the concavity condition mentioned above is necessary in all dimensions.

**Corollary 1.5.** *Let  $n \geq 3$ . There exist a bounded domain  $\Omega \subset \mathbb{R}^n$  and a point  $x_0 \in \Omega$  such that*

$$(1.7) \quad k_\Omega(x, x_0) \leq C d(x, \partial\Omega)^{-\frac{1}{n}} \log \frac{C}{d(x, \partial\Omega)}$$

*for all  $x \in \Omega$  and a quasiconformal mapping  $f : \Omega' \rightarrow \Omega$ , where  $\Omega' \subset \mathbb{R}^n$ , such that  $f$  is not uniformly continuous with respect to the metrics  $d(x, y) = |x - y|$  in  $\Omega$  and  $d_I$  in  $\Omega'$ .*

In Theorem 1.4, the mapping  $f$  actually satisfies

$$(1.8) \quad D_I(f(x'), f(y')) \leq C \left( \log \frac{1}{D_{QH}(x', y')} \right)^{-\frac{1}{s-1}},$$

where  $D_{QH}(x, y) = \text{diam}[x, y]$ . Notice that  $D_{QH}(x', y') \geq D_I(x', y')$  and that  $D_{QH}(x', y') \leq Cd_I(x', y')$  if  $\Omega'$  satisfies a Gehring-Hayman inequality, especially if  $\Omega'$  is Gromov hyperbolic when equipped with the quasihyperbolic metric [2].

One could then hope that Theorem 1.1 extends to hold for all  $s > 1$  under the Gehring-Hayman assumption and even that (1.8) holds for all  $s > 1$ . This turns out not to be the case, even though a weaker version of (1.8) does indeed follow from the results in [7] when  $1 < s < 2$ .

**Theorem 1.6.** *Let  $s \in (2, \infty)$  and  $n \geq 2$ . There exist a bounded domain  $\Omega' \subset \mathbb{R}^n$ , an  $s$ -John domain  $\Omega \subset \mathbb{R}^n$ , both satisfying the Gehring-Hayman inequality, and a quasiconformal mapping  $f : \Omega' \rightarrow \Omega$  such that  $f$  fails to satisfy (1.8).*

It would be interesting to know whether one can allow for  $s = 2$  in Theorem 1.6.

## 2. PROOFS OF THE MAIN RESULTS

*Proof of Theorem 1.2.* Our 2-John domain  $\Omega$  will be constructed inductively as indicated in Figure 1.

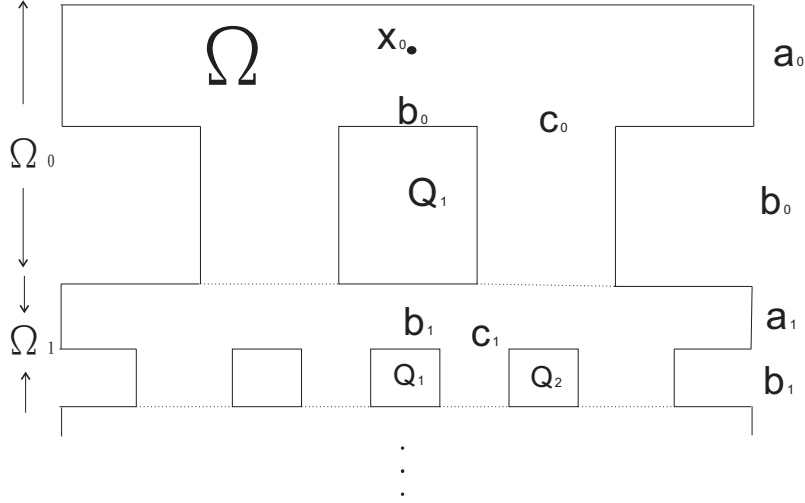


FIGURE 1. the 2-John domain  $\Omega$

Set  $a_j = 2^{-2(j+1)}$ ,  $b_j = 2^{-j}$  and  $c_j = 2^{-2(j+1)}$ . For  $j = 0$ , we let the  $\Omega_0$ -part consist of a rectangle of length 1 and width  $a_0$  and two rectangular “legs” of width  $c_0$  and length  $b_0$ . The two rectangular “legs” are obtained in the following manner: first remove the central square  $Q_1$  of side-length  $b_0$ ; then set the distance between  $Q_1$  and the vertical boundary of  $\Omega_0$  to be  $c_0$ . Next, for  $j = 1$ , we let the  $\Omega_1$ -part consist of a rectangle of length 1 and width  $a_1$  and four rectangular “legs” of width  $c_1$  and length  $b_1$ . The four rectangular “legs” are obtained in a similar fashion as before: first remove 3 squares of side-length  $b_1$ ; then make

them equi-distributed, i.e. the gap between two consecutive squares is  $c_1$ ; finally set the distance between  $Q_2$  and the vertical boundary of  $\Omega_1$  to be  $c_1$ . We continue the process. Let the  $\Omega_j$ -part consist of a rectangle of length 1 and width  $a_j$  and  $2^j$  rectangular “legs” of width  $c_j$  and length  $b_j$ . The rectangular “legs” are obtained by removing  $2^{j+1} - 1$  equi-distributed squares of side-length  $b_j$  in a similar way as before. Among these removed squares, we label from middle to the right-most as  $Q_1, Q_2, \dots, Q_{2^j}$  respectively. According to our construction, the distance between two consecutive removed squares is  $c_j$  and the distance between  $Q_{2^j}$  and the vertical boundary of  $\Omega_j$  is also  $c_j$ . Finally, our domain  $\Omega$  is the union of all  $\Omega_j$ ’s. It is clear from the construction that  $\Omega$  is 2-John and symmetric with respect to the  $y$ -axis.

We next construct our source domain  $\Omega'$  and a quasiconformal mapping  $g : \Omega' \rightarrow \Omega$ , which is not uniformly continuous with respect to the metrics  $d(x, y) = |x - y|$  in  $\Omega$  and  $d_I$  in  $\Omega'$ . Actually, we construct a quasiconformal mapping  $f : \Omega \rightarrow \Omega'$  whose (quasiconformal) inverse has the desired properties.

The idea is demonstrated in Figure 2: we scale the upper part of each  $\Omega_j$  by  $\frac{1}{j}$  and replace the associated  $2^{j+1}$  rectangular “legs” by the same number of new “legs”. The vertical distance between the scaled upper parts of  $\Omega_j$  and  $\Omega_{j+1}$  is set to be  $2j^{-2}$ . We also make the domain  $\Omega'$  symmetric with respect to  $y$ -axis. Since the distance between two consecutive legs in  $\Omega_j$  is  $2^{-j}$ , the distance between the tops of two consecutive “legs” in  $\Omega'_j$  is  $\frac{2^{-j}}{j}$ . For the bottoms, the distance is approximately  $\frac{2^{-j-1}}{j+1}$ .

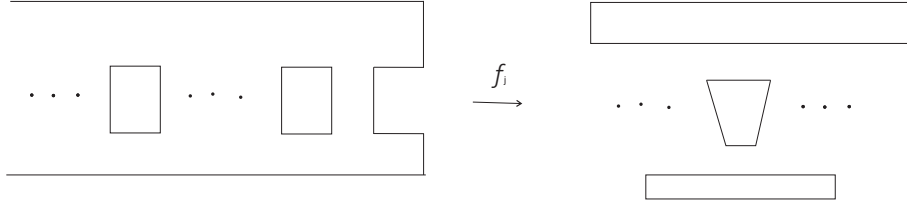


FIGURE 2.  $\Omega$  and  $\Omega'$  in the step  $j$

Recall the labelled squares  $Q_i$ ,  $i = 1, \dots, 2^j$  introduced in  $\Omega_j$ . We denote by  $\tilde{Q}_i$  the “leg” next to  $Q_i$ , on the right. We will construct a quasiconformal mapping  $f_j$  from the (translated) rectangle  $\tilde{Q}_i$  to the (translated) new “leg”  $Q'_i$  as in Figure 3.  $Q'_i$  consists of two parts  $A'$  and  $B'$ . The distance between the bottom line segment  $0\mathbf{a}$  and the top line segment in the  $x$ -direction is

$$m_i^j = \frac{[2^{-j-1} + 2^{-2(j+1)}] \cdot i}{j} - \frac{[2^{-j-1} + 2^{-2(j+1)}] \cdot i}{j+1}.$$

It is clear that  $m_i^j \approx \frac{i \cdot 2^{-j}}{j^2}$  when  $j$  is large. The distance of the top and the bottom in  $y$ -direction is  $\frac{2}{j^2}$ . In Figure 3,  $\mathbf{a} = (\frac{2^{-2(j+1)}}{j+1}, 0)$ ,  $\mathbf{p} = (\frac{i \cdot 2^{-j}}{2j^2}, \frac{1}{j^2})$  and  $\mathbf{q} = (\frac{i \cdot 2^{-j}}{2j^2} + \frac{2^{-(j+1)}}{j}, \frac{1}{j^2})$ . We will write down below a quasiconformal mapping  $f_j : A \rightarrow A'$  such that  $f_j$  maps the bottom line segment of  $A$  linearly to  $0\mathbf{a}$  and the top line segment of  $A$  linearly to  $\mathbf{p}\mathbf{q}$ , respectively. The line  $0\mathbf{p}$  is of the form  $y = k_1x$ , where

$$k_1 = \frac{1/j^2}{i \cdot 2^{-j}/(2j^2)} = \frac{2^{j+1}}{i} \geq 1.$$

Similarly, the line  $\mathbf{a}\mathbf{q}$  is of the form  $y = k_2(x - \frac{2^{-2(j+1)}}{j+1})$ , where

$$k_2 = \frac{1/j^2}{\frac{i \cdot 2^{-j}}{2j^2} + \frac{2^{-(j+1)}}{j} - \frac{2^{-2(j+1)}}{j+1}} \approx \frac{2^j}{i+j}.$$

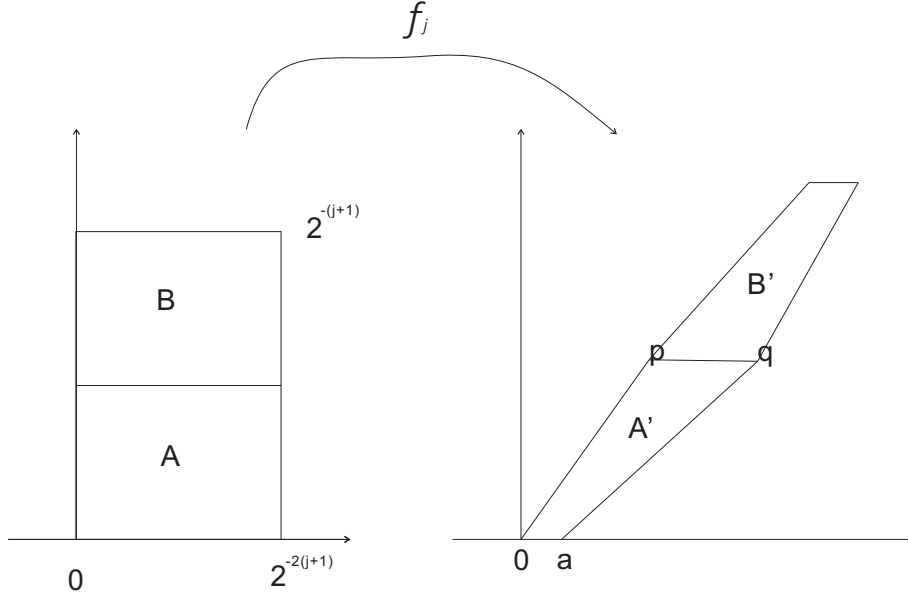


FIGURE 3. The quasiconformal mapping from  $\tilde{Q}_i$  to  $Q'_i$

We are looking for a quasiconformal mapping of the form  $f_j^i(x, y) = (\tilde{g}_j(y)x + g_j(y), k_1g_j(y))$ , where  $\tilde{g}_j(y) = k_1g'_j(y)$  for all  $y \in [0, 2^{-(j+1)}]$  and  $g_j$  is a smooth increasing function. Clearly, such a mapping  $f_j$  maps horizontal line segments to horizontal line segments. We further require that it maps the left side of  $A$  to  $0\mathbf{p}$  and the right side of  $A$  to  $\mathbf{a}\mathbf{q}$ ,  $g_j(0) = 0$ ,  $g_j(2^{-j}) = \frac{1}{j^2}$  and  $\tilde{g}_j(0) = \frac{1}{j+1}$ . By definition,

$$f_j^i(2^{-2(j+1)}, y) = (\tilde{g}_j(y) \cdot 2^{-2(j+1)} + g_j(y), k_1g_j(y)).$$

The further requirements are satisfied if  $\tilde{g}_j = k_1g'_j$ ,

$$(2.1) \quad g_j(y) = k_2 \cdot k_1^{-1} \tilde{g}_j(y) \cdot 2^{-2(j+1)} + \frac{k_2}{k_1} g_j(y) - \frac{k_2}{j} \cdot 2^{-2(j+1)},$$

$$(2.2) \quad g_j(0) = 0, g_j(2^{-j}) = \frac{1}{j^2} \quad \text{and} \quad \tilde{g}_j(0) = \frac{1}{j+1}.$$

One can easily solve the above system of equations by setting  $g_j(y) = a \cdot e^{a_j^i y + c} - b$ , where

$$a_j^i = 2^{2(j+1)} \frac{k_1 - k_2}{k_1 k_2}, \quad b = \frac{1}{k_1(j+1)a_j^i}$$

and the constants  $b$  and  $c$  are chosen such that

$$a \cdot e^c = b \quad \text{and} \quad a \cdot e^{a_j^i 2^{-j} + c} - b = \frac{1}{j^2}.$$

We next show that  $f_j^i$  is a quasiconformal mapping. A direct computation gives us

$$Df_j^i(x, y) = \begin{bmatrix} \tilde{g}_j(y) & \tilde{g}_j'(y)x + g_j'(y) \\ 0 & k_1 g_j'(y) \end{bmatrix}.$$

We only need to show that  $\tilde{g}_j'(y)x + g_j'(y) \leq M k_1 g_j'(y)$ , for some constant  $M$  independent of  $i$  and  $j$ , and for all  $x, y \in A$ . Since  $k_1 \geq 1$ , it suffices to bound  $\tilde{g}_j'(y)x$ . By definition,

$$\tilde{g}_j(y) = k_1 g_j'(y) = k_1 a a_j^i e^{a_j^i y + c}$$

and

$$\tilde{g}_j'(y) = k_1 a_j^i g_j'(y).$$

Hence we only need to find a uniform bound on  $x \cdot a_j^i$ . For this, we first note that  $k_2$  is bounded from below by  $\frac{1}{2}$  and  $\frac{k_1 - k_2}{k_1} \leq 1$ . Since  $x \in [0, 2^{-2(j+1)}]$ , we have

$$a_j^i x \leq \frac{k_1 - k_2}{k_1 k_2} \cdot 2^{2(j+1)} x \leq 2.$$

This implies that  $\tilde{g}_j'(y)x + g_j'(y) \leq 3k_1 g_j'(y)$  and so  $f_j^i$  is quasiconformal. Notice that  $f_j^i(x, 0) = (\frac{x}{j+1}, 0)$ , so that, after suitable translations,  $f_j^i$  matches with our scaling on the top of  $\Omega_{j+1}$ . In a similar manner, one can write down a quasiconformal mapping from  $B$  to  $B'$  such that it coincides with  $f_j^i$  on  $\mathbf{pq}$  and is linear on each line segment. In fact, the quasiconformal mapping just slightly differs from the reflection of  $f_j$  with respect to the line segment  $\mathbf{pq}$  (since the length of  $0\mathbf{a}$  is approximately the same as the length of the top line segment when  $j \rightarrow \infty$  and the picture is exactly a reflection with respect to  $\mathbf{pq}$ ). When a suitable coordinate system is fixed, it is clear that the mappings  $f_j^{i_1}$  and  $f_j^{i_2}$  only differ by a translation in  $x$ -direction and hence the desired global quasiconformal mapping  $f_j$  from  $\Omega_j$  to  $\Omega'_j$  follows by gluing all  $f_j^{i_i}$ 's and the scaling maps.

In this manner, the domain  $\Omega'$  is well-defined. We can define the quasiconformal mapping  $g : \Omega' \rightarrow \Omega$  by setting  $g|_{\Omega'_j} = f_j^{-1}$ . Moreover,

$g$  cannot be uniformly continuous since for each  $j \in \mathbb{N}$ , it maps a rectangle of length  $\frac{1}{j}$  linearly to a rectangle of length 1.  $\square$

*Proof of Corollary 1.3.* Let  $\Omega'$  and  $\Omega$  be the domains given in the proof of Theorem 1.2. Let  $x_0$  be the point marked in Figure 1. It is easy to check that the assumption (1.6) is satisfied and hence the claim follows.  $\square$

*Proof of Theorem 1.4.* We will give the detailed constructions of our domains and quasiconformal mapping for  $n = 3$  and indicate how to pass it to all dimensions at the end of the proof. The idea of the 3-dimensional construction is similar to the one above and we simply fatten the “ $\Omega_0$ ” part of the planar domain in Figure 1 along the third direction; see Figure 4 below.

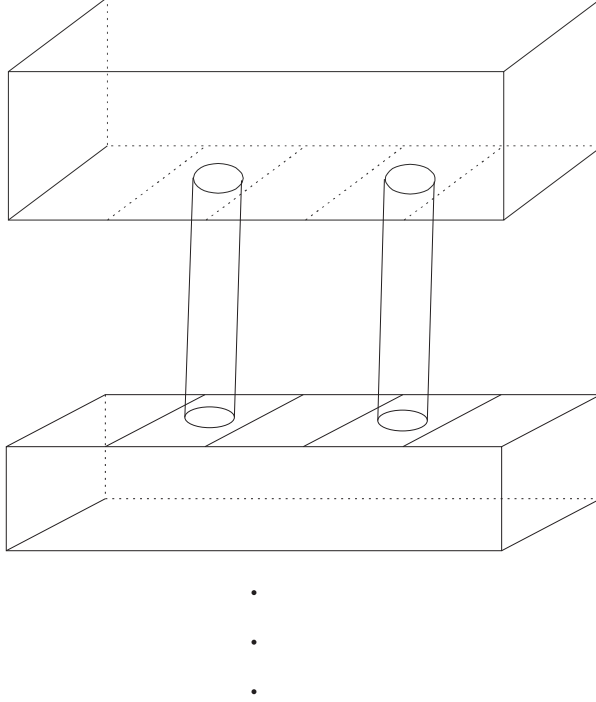


FIGURE 4. The first part of our domain  $\Omega$

The top part of Figure 4 consists of a rectangle of length 1, width  $\frac{1}{2^2}$  and height  $\frac{1}{2^3}$ . In the bottom, the rectangle has length 1, width  $\frac{1}{2^4}$  and height  $\frac{1}{2^6}$ . We attach four cylindrical “legs” of height  $2 \cdot 2^{-2}$  between these rectangles. The radius of the cylinder is about  $2^{-3}$  and the distance between them is about  $2^{-2}$ .

We can proceed our construction in the following manner. At step  $j$ , the top part consists of a rectangle of length 1, width  $2^{-2j}$  and height  $2^{-3j}$ . In the bottom, the rectangle has length 1, width  $2^{-2(j+1)}$  and height  $2^{-3(j+1)}$ . We attach  $2^{2j}$  equi-distributed cylindrical “legs” of



height  $2^{-2j}$  between them. The radius of the cylinder is about  $2^{-3j}$  and the distance between two consecutive cylinders is about  $h_j = j \cdot 2^{-2j}$ . It is clear from our construction that  $\Omega$  is an  $s$ -John domain for any  $s \in (1 + \frac{1}{2}, \infty)$ .

Our source domain  $\Omega'$  is obtained by a similar scaling procedure as in the proof of Theorem 1.2. To be more precise, at step  $j$ , we scale the top rectangle by  $\frac{1}{j^2}$  and replace the associated  $2^j$  cylindrical “legs” by the same number of new “legs”. The vertical distance between the scaled top rectangle and the bottom rectangle is set to be  $h'_j = \frac{2}{j^2}$ .

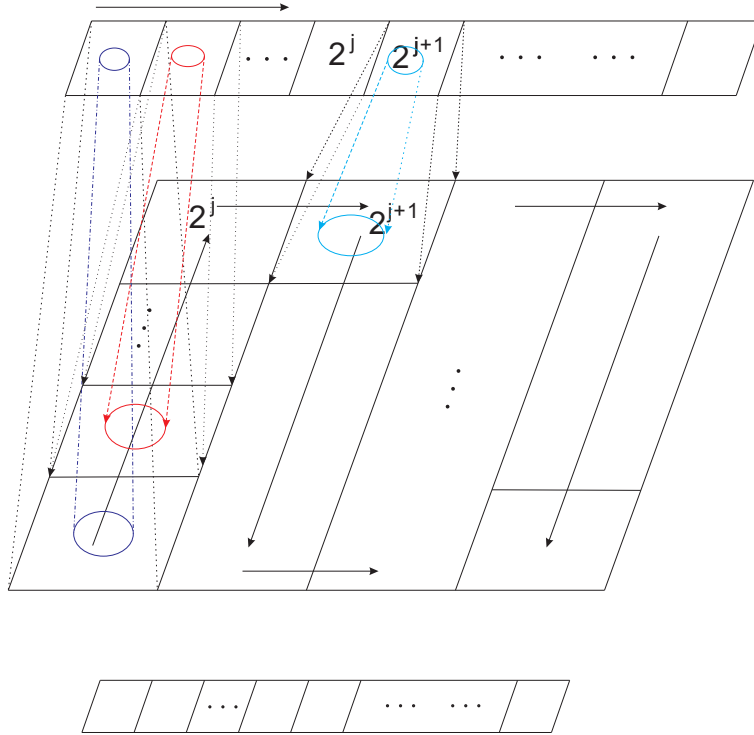


FIGURE 5. The new “legs” at step  $j$

We next explain how to select the new “legs”, see Figure 5 for a top view. In Figure 5, the top rectangle has length  $\frac{1}{j^2}$  and width  $\frac{2^{-2j}}{j^2}$ . It consists of  $2^{2j}$  squares of side-length  $\frac{2^{-2j}}{j^2}$ . The bottom rectangle has length  $\frac{1}{(j+1)^2}$  and width  $\frac{2^{-2(j+1)}}{(j+1)^2}$ . The vertical distance between these rectangles is  $h'_j$ . We insert a square  $S_j$  of side-length  $\frac{1}{j^2}$  in the middle of the two rectangles, i.e. the (vertical) distance between  $S_j$  and either of the rectangles is  $\frac{1}{j^2}$ . We divide  $S_j$  into  $2^{2j}$  subsquares of side-length  $\frac{2^{-j}}{j^2}$ . Next, we set up a one-to-one correspondence between the  $2^{2j}$  squares in the top rectangle and the subsquares in  $S_j$ . To be more precise, we first construct  $2^{2j}$  affine “rectangles” between each square in the top rectangle and each subsquare in  $S_j$  and then we insert a “cylindrical

leg” inside each affine “rectangle”, see Figure 5 for the order of the affine “rectangles”. The radius of the top circle of the “cylindrical leg” is set to be  $\frac{2^{-3j}}{j^2}$  and the radius of the bottom circle is  $\frac{2^{-j}}{j^2}$ . Since the  $2^{2j}$  affine “rectangles” have disjoint interiors, the  $2^{2j}$  “cylindrical legs” are pairwise disjoint. As in the proof of Theorem 1.2, we use a similar construction between  $S_j$  and the bottom rectangle.

Reasoning as in the proof of Theorem 1.2, we only need to write down quasiconformal mappings between these “legs”. Note that our construction implies that all the  $2^{2j}$  “cylindrical legs” are bi-Lipschitz equivalent, with a constant independent of  $j$ . So finally we reduce the problem to the existence of a quasiconformal mapping  $g$  as in Figure 6.

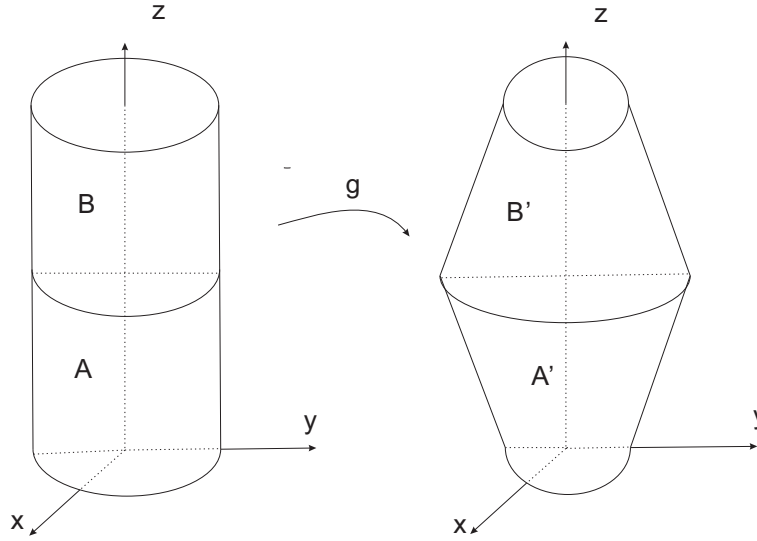


FIGURE 6. The quasiconformal mapping from a “cylinder” to a “double cone”

We will use the coordinate system marked in Figure 6 and write down a quasiconformal mapping  $g$  from  $A$  onto  $A'$  such that  $g$  is a scaling between the bottom disks. Set

$$g(x, y, z) = (g_1(z)x, g_1(z)y, g_2(z)).$$

We require that  $g_1(0) = \frac{1}{j^2}$ ,  $g_1(h_j) = \frac{2^{2j}}{j^2}$ ,  $g_2(0) = 0$ ,  $g_2(h_j) = h'_j$  and  $g'_2(z) = g_1(z)$  for all  $z \in [0, h_j]$ . It is easy to check that with these requirements,  $g$  will be a quasiconformal mapping that maps  $A$  to  $A'$  such that  $g$  is the desired scaling between the bottom disks. One can use the map  $g_2$  of the form  $g_2(z) = a_j(e^{b_j z} - 1)$ , where  $a_j \approx \frac{2^{-2j}}{j^2}$  and  $b_j \approx 2^{2j}$ . We leave the simple verification to the interested readers.

As in the planar case, the global quasiconformal mapping  $f : \Omega' \rightarrow \Omega$  is obtained by gluing all these  $g$ 's and the corresponding scaling mappings. Moreover, reasoning as in the planar case, we can easily conclude that  $f$  cannot be uniformly continuous with respect to the metrics  $d(x, y) = |x - y|$  in  $\Omega$  and  $d_I$  in  $\Omega'$ .

The construction of the general  $n$ -dimensional case can be proceeded in a similar manner. In step  $j$ ,  $\Omega_j$  consists of a  $n$ -dimensional rectangle of length  $a_1 = 1$  and (other) edge-lengths  $a_2 = \cdots = a_{n-1} = 2^{-(n-1)j}$ ,  $a_n = 2^{-nj}$  and  $2^j$  “cylindrical legs” of length  $h_j = j \cdot 2^{-(n-1)j}$ . The radius of the cylinder is  $2^{-nj}$ . So  $\Omega$  is an  $s$ -John domain for any  $s \in (1 + \frac{1}{n-1}, \infty)$ .

The source domain  $\Omega'$  is obtained by a similar scaling procedure as before. To be more precise, at step  $j$ , we scale the top rectangle by  $\frac{1}{j^2}$  and replace the associated  $2^j$  cylindrical “legs” by the same number of new “legs”. The vertical distance between the scaled top rectangle and the bottom rectangle is set to be  $h'_j = \frac{2}{j^2}$ .

We use a similar idea as before to obtain new “legs” between the top rectangle and bottom rectangle as in Figure 5. Namely, we insert a  $(n-1)$ -dimensional cube of edge-length  $\frac{1}{j^2}$  and then divide it into  $2^{(n-1)j}$  subcubes of edge-length  $\frac{2^{-j}}{j^2}$ . Then attach  $2^{(n-1)j}$  affine “rectangles” in a similar manner as before. Inside each affine “rectangle”, we insert a “cylindrical leg”. The radius of the top of the “cylindrical leg” is  $\frac{2^{-nj}}{j^2}$  and the radius of the bottom is  $\frac{2^{-j}}{j^2}$ . Reasoning as before, one essentially only needs to write down a quasiconformal mapping  $g$  between these “legs”.

The global quasiconformal mapping  $f : \Omega' \rightarrow \Omega$  is obtained by gluing all these  $g$ 's and the corresponding scaling mappings. Moreover, reasoning as in the planar case, we can easily conclude that  $f$  cannot be uniformly continuous with respect to the metrics  $d(x, y) = |x - y|$  in  $\Omega$  and  $d_I$  in  $\Omega'$ .

□

*Proof of Corollary 1.5.* Let  $\Omega'$  and  $\Omega$  be the domains given in the proof of Theorem 1.4. Let  $x_0$  be the central point in the first rectangle of  $\Omega$ . It is easy to check that the assumption (1.7) is satisfied and hence the claim follows. □

*Proof of Theorem 1.6.* The idea of the construction is similar to that used in the proof of Theorem 1.2. When  $n = 2$  and  $s > 2$ , our  $s$ -John domain  $\Omega \subset \mathbb{R}^2$  is the same as that in Figure 1 except possible differences in the parameters. When  $n \geq 3$  and  $s > 2$ , we simply fatten the planar picture. In the following, we point out the difference of the construction of the  $s$ -John domain in the planar case and give a more detailed construction of the 3-dimensional analog, while indicating the general construction in the end of the proof.

We first consider the case  $n = 2$ . Fix  $s \in (2, \infty)$ . Let  $\Omega$  be the domain given as in Figure 1 with  $a_j = 2^{-(j+1)}$ ,  $b_j = 2^{-(j+1)}$  and  $c_j = 2^{-(j+1)s}$ . Then  $\Omega$  is an  $s$ -John domain. The domain  $\Omega'$  is constructed in a similar fashion and finally we need a quasiconformal mapping from  $\tilde{Q}_i$  onto  $Q'_i$  as in Figure 3. The only difference from Figure 3 is that

instead of  $\frac{2^{-j}}{j}$ , we set  $d(p, q) = \frac{2^{-j(s-1)}}{j}$ . One can check the desired quasiconformal mapping  $f$  is of the same form as in Figure 3 with the obvious replacement of parameters. Clearly  $f$  is not uniformly continuous with respect to the metrics  $d(x, y) = |x - y|$  in  $\Omega$  and  $d_I$  in  $\Omega'$ .

We next verify that  $\Omega$  satisfies the Gehring-Hayman inequality, i.e. we need to show that

$$(2.3) \quad l([x, y]) \leq Cd_I(x, y)$$

for all  $x, y \in \Omega$ . We first consider the case that  $x, y \in \Omega_j$  for some  $j \in \mathbb{N}$ . Recall that the  $\Omega_j$ -part consists of a rectangle of  $Q_j$  and  $2^j$  rectangular “legs”  $Q_{ji}$ ,  $i = 1, 2, \dots, 2^j$ . If both  $x$  and  $y$  lie in the rectangle  $Q_j$  or both lie in some “leg”  $Q_{ji}$ , then the length of  $[x, y]$  essentially equals to the length of the line segment  $\overline{xy}$  that connects  $x$  and  $y$  and so (2.3) holds. If  $x \in Q_{jk}$  and  $y \in Q_{jl}$ ,  $k, l \in \{1, 2, \dots, 2^j\}$  and  $k < l$ , then  $[x, y]$  can be essentially written as  $\gamma_{xz} \cup \overline{zw} \cup \gamma_{wy}$ , where  $z$  is a point on the (horizontal) core line segment of  $Q_j$  with the same first coordinate as  $x$  such that  $\gamma_{xz}$  is essentially the Euclidean geodesic  $\overline{xz}$  and  $w$  is a point on the (horizontal) core line segment of  $Q_j$  with the same first coordinate as  $y$  such that  $\gamma_{wy}$  is essentially the Euclidean geodesic  $\overline{wy}$ . Since  $d_I(x, y)$  is comparable to the difference of  $x$  and  $y$  in horizontal directions, we easily obtain (2.3). If  $x \in Q_{ji}$  and  $y \in Q_j$ , then  $[x, y]$  can be essentially written as  $\gamma_{xz} \cup \overline{zy}$ , where  $z$  is a point on the (horizontal) line segment of  $Q_j$  with the same first coordinate as  $x$  and the same second coordinate as  $y$  such that  $\gamma_{xz}$  is essentially the Euclidean geodesic  $\overline{xz}$ . Thus (2.3) holds in this case as well. Similar arguments applies for the case when  $x \in \Omega_j$  and  $y \in \Omega_l$  with  $|j - l| = 1$ .

Now we may assume that  $x \in \Omega_j$  and  $y \in \Omega_l$ ,  $j - 1 > l \in \mathbb{N}$ . Consider first the case  $x \in Q_j$  and  $y \in Q_l$ . We may assume that the first coordinate of  $x$  is smaller than or equal to the first coordinate of  $y$ . Let  $Q_{j-1,i}$  be the nearest “leg” (in  $\Omega_{j-1}$ ) to  $x$ . Then the quasihyperbolic geodesic  $[x, y]$  goes through  $Q_{j-1,i}$ , then follows the closest “leg” in  $\Omega_{j-2}$  until it reaches  $Q_l$ , and then essentially goes along the Euclidean geodesic to  $y$ . We may write  $[x, y]$  as  $\cup_{k=l}^j [x, y]_k$ , where  $[x, y]_k = [x, y] \cap \Omega_k$ . Our construction of  $\Omega$  implies that there exists a curve  $\beta_{xy}$  such that  $d_I(x, y) = l(\beta_{xy})$  and that  $|l(\beta_{xy}^k) - l([x, y]_k)| \leq 2^{-k+1}$ ,  $k = l, \dots, j$ , where  $\beta_{xy}^k = \beta_{xy} \cap \Omega_k$ . Notice that

$$l([x, y]) = l_1 + l(\cup_{l < k < j} [x, y]_k) + l_2,$$

where  $l_1 = l([x, y]_j)$  and  $l_2 = l([x, y]_l)$ . It is clear that

$$l(\cup_{l < k < j} [x, y]_k) \approx \sum_{l < k < j} (2^{-ks} + 2^{-k}) \approx 2^{-(l-1)}.$$

Similarly,

$$l(\beta_{xy}) = l'_1 + l(\cup_{l < k < j} \beta_{xy}^k) + l'_2,$$

where  $l'_1 = l(\beta_{xy}^j)$  and  $l'_2 = l(\beta_{xy}^l)$  and

$$l(\cup_{l < k < j} \beta_{xy}^k) \approx \sum_{l < k < j} (2^{-ks} + 2^{-k}) \approx 2^{-(l-1)}.$$

In this case, since  $l_1 < 2^{-(l-1)}$ , we may assume that  $l_2 \geq 2^{-l+2}$ . However, since  $|l_2 - l'_2| \leq 2^{-l+1}$ , we obtain that

$$2l'_2 \geq 2(l_2 - 2^{-l+1}) \geq l_2.$$

Thus (2.3) holds in this case as well. The other cases can be proved via a similar argument. Therefore, we have verified (2.3) in  $\Omega$ .

To verify that  $\Omega'$  satisfies the Gehring-Hayman inequality is more complicated. We first consider the case  $x', y' \in \Omega'_j$  for some  $j \in \mathbb{N}$ . Recall that the  $\Omega'_j$ -part consists of a rectangle of  $Q'_j$  and  $2^j$  cone-like “legs”  $Q'_{ji}$ ,  $i = 1, 2, \dots, 2^j$ . If both  $x'$  and  $y'$  lie in the rectangle  $Q'_j$  or both lie in some “leg”  $Q'_{ji}$ , then the length of  $[x', y']$  essentially equals to the length of the line segment  $\overline{x'y'}$  that connects  $x'$  and  $y'$  and so the Gehring-Hayman inequality holds. If  $x' \in Q'_{jk}$  and  $y' \in Q'_{jl}$ ,  $k, l \in \{1, 2, \dots, 2^j\}$  and  $k < l$ , then there are two different cases for the geodesic  $\gamma'$  that connects  $x'$  and  $y'$  in  $\Omega'$ : either  $\gamma'$  first goes up from  $x'$ , passes through  $Q'_j$  and then goes down to  $y'$  or first goes down from  $x'$ , passes through  $Q'_{j+1}$ , and then goes up to  $y'$ . Essentially we have to deal with two cases, either both  $x'$  and  $y'$  are close to  $Q'_j$  or both  $x'$  and  $y'$  are close to  $Q'_{j+1}$ . For this, we need the following two basic facts: firstly, if  $x', y' \in Q'_j$ , then  $k_{\Omega'}(x', y') = k_{\Omega}(f(x'), f(y'))$ . Secondly, if  $x', y' \in Q'_{ji}$ , then  $k_{\Omega'}(x', y') \approx k_{\Omega}(f(x'), f(y'))$ . For example, if both  $x'$  and  $y'$  are close to  $Q'_j$ , then the quasihyperbolic geodesic  $[x', y']$  has to go up from  $x'$ , pass through  $Q'_j$  and then go down to  $y'$ , since otherwise, the quasihyperbolic distance will be much bigger. If  $x' \in Q'_{ji}$  and  $y' \in Q'_j$ , then  $[x', y']$  goes through  $Q'_{ji}$  to  $Q'_j$  and then essentially follows an Euclidean geodesic to  $y'$ . Thus, the Gehring-Hayman inequality holds. Similar arguments applies for the case when  $x' \in \Omega'_j$  and  $y' \in \Omega'_l$  with  $|j - l| = 1$ .

Now we may assume that  $x' \in \Omega'_j$  and  $y' \in \Omega'_l$ ,  $j-1 > l \in \mathbb{N}$ . Consider first the case  $x' \in Q'_j$  and  $y' \in Q'_l$ . Let  $Q'_{j-1,i}$  be the nearest “leg” (in  $\Omega'_{j-1}$ ) to  $x'$ . Then the quasihyperbolic geodesic  $[x', y']$  goes through  $Q'_{j-1,i}$ , then follow the closest “leg” in  $\Omega'_{j-2}$  until it reaches  $Q'_l$ , and then essentially goes along the Euclidean geodesic to  $y'$ . We may write  $[x', y']$  as  $\cup_{k=l}^j [x', y']_k$ , where  $[x', y']_k = [x', y'] \cap \Omega'_k$ . Our construction of  $\Omega'$  implies that there exists a curve  $\beta_{x'y'}$  such that  $d_I(x', y') = l(\beta_{x'y'})$  and that  $|l(\beta_{x'y'}^k) - l([x', y']_k)| \leq \frac{2}{k^2}$ ,  $k = l, \dots, j$ , where  $\beta_{x'y'}^k = \beta_{x'y'} \cap \Omega'_k$ . Notice that

$$l([x', y']) = l_1 + l(\cup_{l < k < j} [x', y']_k) + l_2,$$

where  $l_1 = l([x', y']_j)$  and  $l_2 = l([x', y']_l)$ . It is clear that

$$l(\cup_{l < k < j} [x', y']_k) \approx \sum_{l < k < j} \left( \frac{1}{k^2} + \frac{2^{-k}}{k} \right) \approx \frac{1}{(l-1)^2}.$$

Similarly,

$$l(\beta_{x'y'}) = l'_1 + l(\cup_{l < k < j} \beta_{x'y'}^k) + l'_2,$$

where  $l'_1 = l(\beta_{x'y'}^j)$  and  $l'_2 = l(\beta_{x'y'}^l)$  and

$$l(\cup_{l < k < j} \beta_{x'y'}^k) \approx \sum_{l < k < j} \left( \frac{1}{k^2} + \frac{2^{-k}}{k} \right) \approx \frac{1}{(l-1)^2}.$$

In this case, since  $l_1 < \frac{2}{(l-1)^2}$ , we may assume that  $l_2 \geq \frac{4}{(l-2)^2}$ . However, since  $|l_2 - l'_2| \leq \frac{2}{l^2}$ , we obtain that

$$2l'_2 \geq 2(l_2 - \frac{2}{l^2}) \geq l_2.$$

Thus the Gehring-Hayman inequality holds in this case as well. The other cases can be proved via a similar argument. Therefore, we have verified the Gehring-Hayman inequality in  $\Omega'$ .

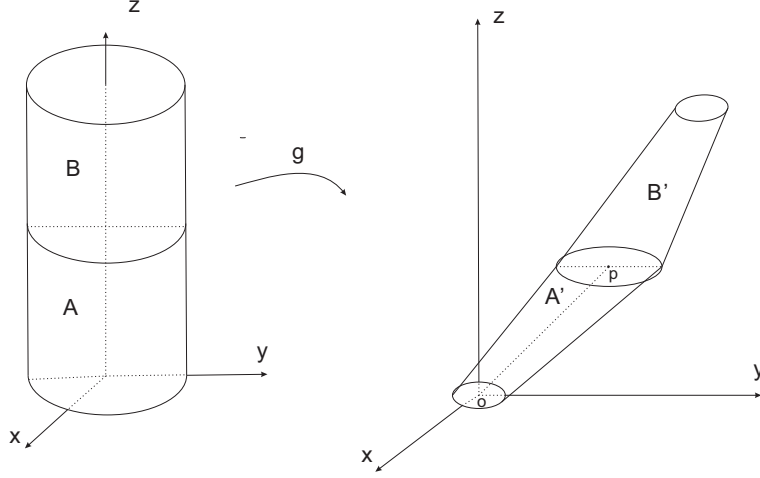
The 3-dimensional construction is similar and we simply fatten the “ $\Omega_0$ ” part of the planar domain in Figure 1 along the third direction; see Figure 4.

The top part of Figure 4 consists of a rectangle of length 1, width  $\frac{1}{2}$  and height  $\frac{1}{2}$ . In the bottom, the rectangle has length 1, width  $\frac{1}{4}$  and height  $\frac{1}{4}$ . We attach two cylindrical “legs” of height  $2^{-1}$  between these rectangles. The radius of the cylinder is about  $2^{-s}$  and the distance between them is about  $2^{-1}$ .

We can proceed our construction in the following manner. At step  $j$ , the top part consists of a rectangle of length 1, width  $2^{-j}$  and height  $2^{-j}$ . In the bottom, the rectangle has length 1, width  $2^{-j-1}$  and height  $2^{-j-1}$ . We attach  $2^j$  equi-distributed cylindrical “legs” of height  $2^{-j}$  between them. The radius of the cylinder is about  $2^{-js}$  and the distance between two consecutive cylinders is about  $2^{-j}$ . It is clear from our construction that  $\Omega$  is an  $s$ -John domain.

Our source domain  $\Omega'$  is obtained by a similar scaling procedure as in the proof of Theorem 1.2. To be more precise, at step  $j$ , we scale the top rectangle by  $\frac{1}{j}$  and replace the associated  $2^j$  cylindrical “legs” by the same number of new “legs”. The vertical distance between the scaled top rectangle and the bottom rectangle is set to be  $\frac{2}{j^2}$ .

The new “legs” are obtained by rotating the corresponding new “legs” in Figure 3 along the third direction, see Figure 7. Reasoning as in the proof of Theorem 1.2, we only need to write down a quasiconformal mappings between these “legs”. For this, we use the coordinate system

FIGURE 7. A quasiconformal mapping  $g$  between “legs”

marked in Figure 7 and look for  $g$  of the form

$$g(x, y, z) = (g_1(z)x, g_1(z)y + g_2(z), g_3(z)).$$

We require that  $g_1(0) = \frac{1}{j}$ ,  $g_2(0) = g_3(0) = 0$ ,  $g_1(2^{-j}) = \frac{2^j}{j}$ ,  $g_2(2^{-j}) \approx \frac{1}{2j^2}$  and  $g_3(2^{-j}) = \frac{1}{2j^2}$ . We can take

$$g_1(z) = a_j b_j e^{b_j z}, \quad g_2(z) \approx g_3(z) = a_j (e^{b_j z} - 1),$$

where  $a_j \approx 2^{-j} \cdot j^{-2}$  and  $b_j \approx j \cdot 2^j$ . It is easy to check that  $g$  is a quasiconformal mapping with the desired property and we leave the detailed verification to the interested author.

As in the planar case, the global quasiconformal mapping  $f : \Omega' \rightarrow \Omega$  is obtained by gluing all these  $g$ 's and the corresponding scaling mappings. Moreover, reasoning as in the planar case, we can easily conclude that  $f$  cannot be uniformly continuous with respect to the metrics  $d(x, y) = |x - y|$  in  $\Omega$  and  $d_I$  in  $\Omega'$ . Following the arguments used in the planar case, we easily deduce that both  $\Omega'$  and  $\Omega$  satisfies the Gehring-Hayman inequality.

The construction of the general  $n$ -dimensional case can be proceeded in a similar manner. In step  $j$ ,  $\Omega_j$  consists of a  $n$ -dimensional rectangle of length  $a_1 = 1$  and (other) side-lengths  $a_2 = \dots = a_n = 2^{-j}$  and  $2^j$  “cylindrical legs”. The radius of the cylinder is  $2^{-js}$ . So  $\Omega$  is an  $s$ -John domain. The source domain  $\Omega'$  is obtained by a similar scaling procedure as before. To be more precise, at step  $j$ , we scale the top rectangle by  $\frac{1}{j}$  and replace the associated  $2^j$  cylindrical “legs” by the same number of new “legs”. The vertical distance between the scaled top rectangle and the bottom rectangle is set to be  $\frac{2}{j^2}$ .

The new “legs” are obtained by rotating the corresponding new “legs” in Figure 3 along the  $n$ -th direction, see Figure 7. Reasoning as in the proof of Theorem 1.2, we only need to write down a quasiconformal

mappings between these “legs”. We leave the remaining verifications to the interested readers.  $\square$

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